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The Lattice of $\mathscr{J}$-Classes of $(\mathscr{J}, \sigma)$-Irreducible Monoids

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In this paper, we introduce a new concept, namely, the $(\mathscr{J}, \sigma)$-*irreducible monoid*. Let G_0 be a simple algebraic group. Let σ be a surjective endomorphism of G_0 such that $(G_0)_\sigma$, the fixed points of G_0 under σ, is finite. We construct a $(\mathscr{J}, \sigma)$-irreducible monoid M with G the unit group. Extending σ to M, we obtain a finite monoid M_σ which is $\mathscr{J}$-irreducible. If M is $\mathscr{J}$-irreducible, we give a precise recipe to determine the lattices of $\mathscr{J}$-classes of M and M_σ.

1. PRELIMINARIES

1.1. *An algebraic monoid* is an affine algebraic variety M defined over an algebraically closed field K together with an associative morphism m: $M \times M \to M$ and a two-sided unit $1 \in M$ for m. M is an *irreducible algebraic monoid* if M is irreducible as a variety. $E(M) = \{e \in M \mid e^2 = e\}$ is the set of idempotents of M. $G = G(M) = \{x \in M \mid \exists \ y \in M$ s.t. $xy = yx = 1\}$ is the unit group of M. An irreducible monoid M is *reductive* if its unit group G is reductive. An irreducible monoid M with 0 is *semisimple* if G is reductive and $\dim ZG = 1$.

$(*)$ Throughout this paper, the monoids are always reductive.

1.2. For M and G as above, let T be some maximal torus of G and let $B \supset T$ be the Borel subgroup of G. $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ is the fundamental root system with respect to B and T, and Φ (Φ^+ or Φ^-) is the root (positive or negative) system. $W = N_G(T)/T$ is the Weyl group, which is isomorphic to the group generated by the set of simple reflections $S = \{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_l}\}$.

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1.3. Green's $\mathcal{J}$-relation on M is defined by $a\mathcal{J}b$ if $MaM = MbM$. The equivalence classes for this relation are called *$\mathcal{J}$-classes*. It is easy to see that $MaM = MbM$ if and only if $GaG = GbG$ and in this case the $\mathcal{J}$-class of a is GaG. If we consider the group $G \times G$ acting on M by $(g, h) \cdot a = gah^{-1}$ for $g, h \in G$ and $a \in M$, then the J-class of a is the orbit GaG. We define a partial order on the set of $\mathcal{J}$-classes by $MaM \le MbM$ iff $MaM \subseteq MbM$. Define $\mathcal{U}(M) = \{J \text{ some } \mathcal{J}\text{-class} \mid E(J) \neq \emptyset\}$. $\mathcal{U}(M)$ is the set of $\mathcal{J}$-classes of M which contain idempotents. It inherits a partial order from the set of all $\mathcal{J}$-classes. By the assumption of $(*)$, it follows that $\mathcal{U}(M)$ equals the set of all $\mathcal{J}$-classes on M [11, Proposition 3.6]. From the semigroup theoretic point of view, one interesting problem is to describe $\mathcal{U}(M)$. By [5, Corollaries 6.8 and 6.10, Theorem 6.25] the problem of determining $\mathcal{U}(M)$ is to find the set of W-conjugacy classes in $E(\overline{T})$. Note that $E(\overline{T})$ is a finite, relatively complemented lattice with partial order $e < f$ defined by $ef = fe = e \neq f$ [5, Theorem 6.20].

1.4. Let M be an irreducible monoid with unit group G. Then $\Lambda \subseteq E(\overline{T})$ is a *cross-section lattice* if

(1) $|\Lambda \cap J| = 1$ for all $J \in \mathcal{U}(M)$.

(2) If $e, f \in \Lambda$, then $J_e \ge J_f$ implies $e \ge f$.

The cross-section lattice is a sublattice of $E(\overline{T})$, and $\Lambda = \{e \in E(\overline{T}) \mid B \subseteq C_G^r(e)\}$ [5, Theorem 9.10]. $C_G^r = \{x \in G \mid ge = ege\}$ is the *right centralizer* of e. So finding $\mathcal{U}(M)$ is equivalent to finding Λ.

2. $\mathcal{J}$-IRREDUCIBLE MONOIDS

2.1. A reductive monoid M is a *$\mathcal{J}$-irreducible monoid* if all minimal nonzero idempotents are conjugate.

To construct a $\mathcal{J}$-irreducible monoid, we choose an algebraic group G_0 and a rational representation $\rho\colon G_0 \to GL_n$ for some n, with finite kernel. Then $M(\rho) = \overline{K^*\rho(G_0)}$ is an irreducible monoid but not necessarily $\mathcal{J}$-irreducible. However, $M(\rho)$ is $\mathcal{J}$-irreducible if ρ is an irreducible representation and G_0 is semisimple (see [8, Corollary 8.3.3]). Putcha and Renner [6, 4.16] gave a precise recipe for constructing the lattice of $\mathcal{J}$-classes and the type map of a $\mathcal{J}$-irreducible monoid M. We sketch the main results here. For $e \in \Lambda$, let

$$I(e) = \{\alpha \in \Delta \mid s_\alpha e = es_\alpha \neq e\}$$

and

$$J(e) = \{\alpha \in \Delta \mid s_\alpha e = es_\alpha = e\}.$$

Let e_0 be the unique minimal nonzero idempotent in Λ, and define $J_0 = J(e_0)$.

THEOREM 2.2 (Putcha and Renner [6, 4.11, 4.12, 4.16]). (1) $J(e) = \{\alpha \in J_0 \setminus I(e) \mid s_\alpha s_\beta = s_\beta s_\alpha$ *for all* $\beta \in I(e)\}$.

(2) $I\colon \Lambda \to \mathscr{P}(\Delta)$, $e \to I(e)$, *is an order-preserving injection.*

(3) $S(\subseteq \Delta) = I(e)$ *for some* $e \in \Lambda$ *iff no component of* S *lies entirely in* J_0.

PROPOSITION 2.3. *Let* ρ *be the irreducible representation corresponding to a dominant weight* $\mu = \sum_{i=1}^{l} a_i \mu_i$. *Let* $I(\mu) = \{\alpha_i \mid a_i \neq 0\}$. *Then* $J_0 = \Delta \setminus I(\mu)$.

Proof. Let $v \in V$ be the highest weight vector of weight μ. Then, by [2, Section 31], the isotropic group of Kv is $P_L = BW_L B$, where $L = \{\alpha \in \Delta \mid \rho(s_\alpha)v = v\} = \Delta \setminus I(\mu)$. So it suffices to show that $L = J_0$. Let Γ be the set of weights of the representation ρ. Let $V_\lambda = \{v \in V \mid \rho(t)v = \langle \lambda, t \rangle v = \lambda(t)v\}$ for $\lambda \in \Gamma$. Then $V = \bigoplus_{\lambda \in \Gamma} V_\lambda$. Note that $V_\mu = Kv$. To simplify the notation, we denote $\rho(g)x$ by $g \cdot x$ for $g \in G$ and $x \in V$. Now, by [6, Theorem 5.3], $Kv = e_0 \cdot V$, since $Kv = V^{B_u}$. So

$$e_0 \cdot v = \langle \mu, e_0 \rangle v = v,$$

since $\langle \mu, e_0 \rangle \neq 0$ and e_0 is an idempotent. Also

$$e_0 \cdot v_\lambda = 0 \qquad \text{for } v_\lambda \in V_\lambda \text{ and } \mu \neq \lambda \in \Gamma.$$

Then

$$\begin{aligned} e_0 \cdot v = e_0 s_\alpha \cdot v &= s_\alpha s_\alpha^{-1} e_0 s_\alpha \cdot v \\ &= s_\alpha \cdot \langle \mu, s_\alpha^{-1} e_0 s_\alpha \rangle v \\ &= s_\alpha \cdot \langle s_\alpha \mu, e_0 \rangle v = s_\alpha \cdot \langle \mu, e_0 \rangle v \\ &= s_\alpha e_0 \cdot v \qquad \text{for } \alpha \in L, \end{aligned}$$

and

$$\begin{aligned} 0 = e_0 \cdot v_\lambda &= s_\alpha e_0 \cdot v_\lambda \\ e_0 s_\alpha \cdot v_\lambda &= s_\alpha s_\alpha^{-1} e_0 s_\alpha \cdot v_\lambda \\ &= s_\alpha \cdot \langle \lambda, \left(s_\alpha^{-1} e_0 s_\alpha\right) \rangle v_\lambda \\ &= s_\alpha \cdot \langle s_\alpha \lambda, e_0 \rangle v_\lambda \\ &= 0 \qquad \text{for } \alpha \in L \text{ and } \mu \neq \lambda \in \Gamma, \end{aligned}$$

since $s_\alpha \lambda < \mu$. Thus $e_0 = s_\alpha e_0 = e_0 s_\alpha$, so that $L \subseteq J_0$.

Conversely, for $\alpha \in J_0$,

$$v = e_0 \cdot v = s_\alpha e_0 \cdot v = e_0 s_\alpha \cdot v.$$

Then

$$s_\alpha \cdot v = v.$$

Thus $\alpha \in L$, so that $J_0 \subseteq L$. ∎

3. IDEMPOTENTS AND CONVEX HULL

3.1. Using the theory of torus embedding, Solomon [11, Section 5] gives a detailed description about how to compute idempotents in $E(\overline{T})$ and the cross-section lattices Λ. The following result is due to the second named author (see [11, Section 5] and [8, Section 3]).

PROPOSITION 3.2. *Let M be the monoid as constructed in* 2.1 *by a rational representation ρ. Let $C^\vee(1)$ be a convex hull of the set $\Phi(\rho)$ of weights of ρ. Then*

(1) *There is a lattice isomorphism from the face lattice of $\mathscr{F}(C^\vee(1))$, including the empty face, to $E(\overline{T})$.*

(2) *There is a lattice isomorphism from the set of W-orbits of $\mathscr{F}(C^\vee(1))$ to the cross-section lattice $\Lambda \backslash 0$.*

(3) *If ρ corresponds to a dominant weight μ, then $C^\vee(1)$ is the convex hull of $W \cdot \mu$.*

4. $(\mathscr{J}, \sigma)$-IRREDUCIBLE MONOIDS

4.1. A $(\mathscr{J}, \sigma)$-*irreducible* monoid M is a reductive monoid with an endomorphism σ of M such that (W, σ) acts transitively on $E_1(\overline{T})$, the set of minimal nonzero idempotents in $E(\overline{T})$. More precisely, let $\Lambda_1 = \Lambda \cap E_1(\overline{T})$. Then σ permutes the set Λ_1. In other words, for any $e \in E_1(\overline{T})$,

$$E_1(\overline{T}) = W \cdot e \cup W \cdot \sigma(e) \cup W \cdot \sigma^2(e) \dots .$$

Obviously, M is a $(\mathscr{J}, \sigma)$-irreducible monoid if M is $\mathscr{J}$-irreducible and $\sigma = 1$. This special case is well studied in [6]. However, most of the $(\mathscr{J}, \sigma)$-irreducible monoids are usually not $\mathscr{J}$-irreducible.

The definition of a $(\mathscr{J}, \sigma)$-irreducible monoid is motivated by the second named author's construction of finite reductive monoids as in [7, Section 4] or [9].

Next we construct a class of $(\mathscr{J}, \sigma)$-irreducible monoids such that $M_\sigma = \{x \in M \mid \sigma(x) = x\}$ is finite. Let G_0 be a simple algebraic group. Chevalley [12, Section 11] classified all endomorphisms $\sigma \colon G_0 \to G_0$ such that $G_{0\sigma} = \{x \in G_0 \mid \sigma(x) = x\}$ is a finite group. Let $\sigma\colon G_0 \to G_0$ be a surjective morphism such that $G_{0\sigma}$ is finite. Then there is a Borel subgroup B_0 of G_0 and a maximal torus T_0 of B_0 such that $\sigma(T_0) = T_0$ and $\sigma(B_0) = B_0$ [12, Section 10]. Let Δ be the fundamental root system associated with the maximal torus T_0 of G_0. Let $\sigma^*\colon X(T_0) \to X(T_0)$ be the map induced by $\sigma|_{T_0}$. Then there exists a nontrivial symmetric automorphism of the Dynkin diagram ϱ and a scalar factor $q\colon \Phi \to \mathbf{N}$ such that

$$\sigma^*(\varrho(\alpha)) = q(\alpha)\alpha \qquad \text{for all } \alpha \in \Delta. \tag{4.1.1}$$

There are two types of σ.

(1) *Chevalley and Steinberg type*: If the root systems are A_n, D_n, and E_6, then $\sigma = Fr_q \circ \varrho$, where $Fr_q(t) = t^q$ for all $t \in T$ and q is some power of p, the characteristic of the field K.

(2) *Suzuki and Ree type*: If the root system is C_2, G_2, or F_4, then $q(\alpha) = p^a$ for α long, and $q(\alpha) = p^{a+1}$ for α short. Here $p = 2$ for C_2 and F_4, and $p = 3$ for G_2.

4.2. Suppose σ is of Steinberg type. Choose a dominant weight μ in $X(T)$. Let $\rho_i\colon G_0 \to GL(V_i)$ be the irreducible representations corresponding to $\varrho^{i-1}(\mu)$ for $i = 1, 2$ (or 3 for D_4 only).

Define

$$\rho = \bigoplus \rho_i\colon G_0 \to GL\Big(\bigoplus V_i\Big) \tag{4.2.1}$$

and $M = M(\rho) = \overline{K^*\rho(G_0)}$.

THEOREM 4.3. (1) *$M(\rho)$ is a $(\mathscr{J}, \sigma)$-irreducible monoid.*

(2) *M_σ is finite.*

(3) *M_σ is $\mathscr{J}$-irreducible.*

We leave the proof for later. Let $T = K^*\rho(T_0)$ and $G = K^*\rho(G_0)$. Since ρ has a finite kernel, $\rho(G_0)$ is also a simple algebraic group. We can identify the root system of $\rho(G_0)$ relative to $\rho(T_0)$ and $\rho(B_0)$ with that of G_0. Also there is a commutative diagram:

$$\begin{array}{ccc} G_0 & \xrightarrow{\sigma} & G_0 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ \rho(G_0) & \longrightarrow & \rho(G_0) \end{array} \tag{4.3.1}$$

We use the same notation σ for the bottom map.

Since $X(T) = X(\rho(T_0)) \oplus \mathbf{Z}$, let $(\mu, 1) \in X(T)$ and σ^* extends to $X(T)$ by $\sigma_1^*(\chi, t) = (\sigma^*(\chi), t^q)$. Then σ_1^* is induced from $\sigma_1\colon G \to G$, $\sigma_1(g, t) = (\sigma(g), t^q)$. Note that $\sigma^*(\mu) = q\mu_2$.

Let $C_i^\vee$ be the smallest convex cone containing $(W \cdot \varrho^{i-1}(\mu), 1)$. Let $C^\vee$ be the smallest convex cone containing $\bigcup C_i^\vee$. Then $M(\rho)$ is the semisimple monoid corresponding to the *polyhedral root system* $(\Phi, X(T), C^\vee = X(\overline{T}))$ (see [8]).

LEMMA 4.4 (Renner [10]). *σ_1: $G \to G$ extends to a surjective morphism σ_1: $M \to M$ such that G_{σ_1} and M_{σ_1} are finite iff*

(1) $\sigma_1^*(C^\vee) \subseteq C^\vee$, *and*

(3) $nC^\vee \subseteq \sigma_1^*(C^\vee)$.

4.5. *Proof of Theorem* 4.3. For (1), it is clearly true by the construction.

For (2), it follows since $C^\vee$ satisfies (1) and (2) of Lemma 4.4. Changing notation, let $\sigma = \sigma_1$ denote the finite morphism σ: $G \to G$ and σ: $M \to M$ such that G_σ and M_σ are finite. So M_σ is the finite monoid with unit group G_σ.

For (3), it is enough to show the case $\varrho^2 = 1$. Let e_1, e_2 be two nonzero minimal idempotents in the cross-section lattice Λ of M. Then $\sigma(e_1) = e_2$ and $\sigma(e_2) = e_1$. So

$$\sigma(e_1 \vee e_2) = \sigma(e_1) \vee \sigma(e_2) = e_1 \vee e_2 \in \Lambda_\sigma, \tag{4.5.1}$$

where $\Lambda_\sigma = \{e \in \Lambda \mid \sigma(e) = e\}$ is the cross-section lattice of M_σ. For any $e \in \Lambda_\sigma \setminus 0$, either $e > e_1$ or $e > e_2$. In the former case, $e = \sigma(e) > \sigma(e_1) = e_2$, so that $e > e_1 \vee e_2$. Then $e_1 \vee e_2$ is the unique nonzero minimal idempotent in Λ_σ. ∎

4.6. For σ of Suzuki or Ree type, one cannot obtain a finite monoid M_σ from a semisimple monoid M as in [7, Section 4], because the requirement that σ: $M \to M$ would imply that $\sigma^2(t) = t^{p^{2a+1}}$ for all $t \in Z(G(M))^0$ and some $a > 0$ [12]. But that $\sigma(t) = t^m$ implies $\sigma^2(t) = t^{m^2} = t^{p^{2a+1}}$ for all t. That is impossible. The following construction is contained in [7].

Assume G_0 is a simple algebraic group of type C_2, F_4, or G_2, and σ: $G_0 \to G_0$ is an endomorphism of Ree or Suzuki type. Assuming $\sigma(T_0) = T_0$ and $\sigma(B_0) = B_0$ with $T_0 \subseteq B_0$, we can arrange the simple roots $\Delta = \{\alpha_i, \beta_i\}_{i=1}^s$ ($s = 1$ or 2) for α_i short and β_i long so that $\sigma^*(\alpha_i) = p^a\beta_i$ and $\sigma^*(\beta_i) = p^{a+1}\alpha_i$. Choose $\mu = \sum_{i=1}^{2s} a_i \mu_i$ (some a_i may be 0, not as in [7]), a dominant weight. Then $\sigma^*(\mu)$ is also a dominant weight, and $\sigma^*(\mu)$ is not a multiple of μ since σ^* has no rational eigenvalues on $X(T) \otimes \mathbf{R}$. Let $\mu' = \sigma^*(\mu)/p^a$. Let ρ_1: $G \to Gl(V_1)$ and ρ_2: $G \to Gl(V_2)$ be the irreducible representations with highest weight μ and μ', respectively, and let $\rho = \rho_1 \oplus \rho_2$. Define

$$M \text{ to be the normalization of } \overline{\rho(G_0)(K^* \times K^*)}.$$

Let $G = \overline{\rho(G_0)(K^* \times K^*)}$, and let T be the maximal torus of G containing T_0. Then $X(T) = X(T_0) \oplus \mathbf{Z}^2$. Let $\chi = (\mu, 1, 0)$ and $\chi' = (\mu', 0, 1)$. Then $C^\vee = X(\overline{T})$ is the smallest polyhedral cone containing $W \cdot \chi \cup W \cdot \chi'$. The morphism $\sigma\colon G_0 \to G_0$ extends to $\sigma\colon G \to G$ by $\sigma(g, s, t) = (\sigma(g), t^{p^{a+1}}, s^{p^a})$ and this yields $\sigma^*\colon X(T) \to X(T)$ by $\sigma^*(\gamma, \alpha, \beta) = (\sigma^*(\gamma), p^{a+1}\beta, p^a\alpha)$. Since $\sigma^*(\chi) = p^a\chi'$ and $\sigma^*(\chi') = p^{a+1}\chi$, $\sigma^*(X(\overline{T})) \subseteq X(\overline{T})$ with $p^{2a+1}X(\overline{T}) \subseteq \sigma^* X(\overline{T})$. Thus there exists a unique $\sigma\colon M \to M$ extending $\sigma\colon \overline{T} \to \overline{T}$ and $\sigma\colon G \to G$ by Lemma 4.4.

PROPOSITION 4.7 [7, Theorem 4.10]. *$M(\rho)$ is $(\mathscr{J}, \sigma)$-irreducible and M_σ is finite and $\mathscr{J}$-irreducible.*

4.8. We are now in a position to ask the following questions.

(A) What is $\mathscr{U}(M)$ or, equivalently, Λ?

(B) What is $\mathscr{U}(M_\sigma)$ or, equivalently, Λ_σ?

If $\mu = \sum_{i=1}^{l} a_i \mu_i$, define $I(\mu) = \{\alpha_i \in \Delta \mid a_i \neq 0\}$. More precisely, we should answer these two questions in terms of the following separate cases:

(1) $\sigma^*(\mu) = q^n(\mu)$.

(2) $\sigma^*(\mu) \neq q^n(\mu)$.

(a) $I(\mu) = I(\sigma^*(\mu))$.

(b) $I(\mu) \neq I(\sigma^*(\mu))$.

PROPOSITION 4.9 ([11, 5.49], [6, Section 4], and [8, 3.5]). *For Steinberg type, let $C_i^\vee(1)$ be the convex hull of $W \cdot \rho^{i-1}\mu$. Let $C^\vee(1)$ be the convex hull of $\bigcup C_i^\vee(1)$. For Suzuki or Ree type, let $C_1^\vee(1)$ be the convex hull of $W \cdot (\mu, 1, 0)$. Let $C_2^\vee(1)$ be the convex hull of $W \cdot (\mu', 0, 1)$. Let $C^\vee(1)$ be the convex hull of $C_1^\vee(1)$ and $C_2^\vee(1)$. Then*

(1) *There is a lattice isomorphism from the face lattice of $\mathscr{F}(C^\vee(1))$, including the empty face, to $E(\overline{T})$.*

(2) *There is a lattice isomorphism from the set of W-orbits of $\mathscr{F}(C^\vee(1))$ to the cross-section lattice $\Lambda \setminus 0$.*

COROLLARY 4.10. *If σ is of Steinberg type and $\sigma^*(\mu) = q(\mu)$, then M is $\mathscr{J}$-irreducible and $\mathscr{U}(M) \cong \mathscr{U}(M(\rho_1))$, where $M(\rho_1) = \overline{K^*\rho_1(G_0)}$.*

Proof. This follows immediately from (2) of proposition 4.9 and the fact that $C^\vee(1) = C_i^\vee(1)$ for all i. ∎

Let Δ_ϱ be the set of equivalence classes under ϱ, i.e., the set of ϱ-orbits of Δ. It can be considered as the fundamental root system relative to B_σ and T_σ [1, Section 13]. More precisely, we identify Δ_ϱ case by case as follows:

(1) For case A_l, let $l = 2k - 1$ and $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Then Δ_ϱ is

$$(\alpha_1 + \alpha_{2k-1})/2, \dots, (\alpha_{k-1} + \alpha_{k+1})/2, \alpha_k$$

of type C_k.

(2) For case A_l, let $l = 2k$ and $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Then Δ_ϱ is

$$(\alpha_1 + \alpha_{2k})/2, \dots, (\alpha_k + \alpha_{k+1})/2$$

of B_k.

(3) For case D_l, let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and $\varrho(\alpha_{l-1}) = \alpha_l$. Then Δ_ϱ is

$$\alpha_1, \alpha_2, \dots, \alpha_{l-2}, (\alpha_{l-1} + \alpha_l)/2$$

of type B_{l-1}.

(4) For case D_4, let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Let $\varrho^3 = 1$ and let α_1 be the root in the center of the Dynkin diagram of D_4. Then Δ_ϱ is

$$\alpha_1, (\alpha_2 + \alpha_3 + \alpha_4)/3$$

of type G_2.

(5) For case E_6, let $\Delta = \{\alpha_1, \dots, \alpha_6\}$. Let $\varrho(\alpha_i) = \alpha_{7-i}$ for $i = 1, 2$. Then Δ_ϱ is

$$(\alpha_1 + \alpha_6)/2, (\alpha_2 + \alpha_5)/2, \alpha_3, \alpha_4$$

of type F_4.

Define a natural projection $\pi\colon \Delta \to \Delta_\varrho$ by

$$\pi(\alpha) = \frac{\sum_{i=1}^{m} \rho^{i-1}(\mu)}{m},$$

where m is the order of ϱ. Let μ and σ be as in Corollary 4.10. Let $I'(\mu) = \pi(I(\mu))$. Since $\varrho(I(\mu)) = I(\mu)$, then $\pi^{-1}(I'(\mu)) = I(\mu)$. Let $J_0' = \Delta_\varrho \setminus I'(\mu)$. Since $\varrho(I(\mu)) = I(\mu)$, then $\pi^{-1}(I'(\mu)) = I(\mu)$. Thus $J_0' = \pi(J_0)$. We have

THEOREM 4.11. (1) *There is an order-preserving injection*

$$e \in \Lambda_\sigma \to K(e) \subseteq \mathscr{P}(\Delta_\varrho).$$

(2) *Any* $S(\subseteq \Delta_\varrho) = K(e)$ *for some* $e \in \Lambda_\sigma$ *if and only if* S *has no connected component completely contained in* J_0'.

Proof. (1) For $e \in \Lambda$, according to Theorem 2.2, there is a subset $I(e) \subseteq \Delta$ such that no connected components of $I(e)$ lie in J_0 by Theorem 2.2. Furthermore, $e \in \Lambda_\sigma$ if and only if $\varrho(I(e)) = I(e)$. Let $K(e) = \pi(I(e))$. This proves (1).

(2) The above proof implies the "if" part. So it suffices to show the "only if" part. That is, if $S(\subseteq \Delta_\varrho)$ has no connected components contained in J_0', then $S = K(e)$ for some $e \in \Lambda_\sigma$.

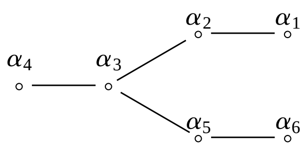
Let $I = \pi^{-1}(S)$. Then $\varrho I = I$ by the definition of π. It is enough to show that I has no connected components contained in J_0.

Suppose that I has a connected component I_1 contained in J_0. Then $\varrho(I_1)$ is also a connected component of I, and $\varrho(I_1) \subseteq J_0$ since $\varrho(J_0) = J_0$. Thus $S_1 = \pi(I_1) \subseteq \pi(J_0) = J_0'$, so that S_1 being a connected component of S is contained in J_0'. But that is a contradiction. ∎

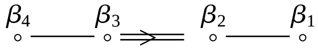
5. CASE E_6^2

5.1. Using Theorems 2.2 and 4.11, we give a complete list of the Hasse diagrams of $\Lambda\backslash 0$ and $\Lambda_\sigma\backslash 0$ for the $\mathscr{J}$-irreducible monoids M and M_σ, respectively, associated with type E_6^2.

5.2. Let the Dynkin diagram of E_6 be as follows:



Then the Dynkin diagram for Δ_ϱ is



where $\beta_4 = \alpha_4$, $\beta_3 = \alpha_3$, $\beta_2 = (\alpha_2 + \alpha_5)/2$, $\beta_1 = (\alpha_1 + \alpha_6)/2$. We assume that k, k_i for $i = 1, 2, \ldots$ are positive integers in the following.

(1) $\mu = k\mu_4$. In this case $J_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ and $J_0^* = \{\beta_1, \beta_2, \beta_3\}$. $\Lambda\backslash 0$ is shown in Fig. 1, where $i_1 i_2 \ldots$ represents the subset $\{\alpha_{i_1}, \alpha_{i_2}, \ldots\}$ of Δ in the Hasse diagrams of $\Lambda\backslash 0$, and $\overline{i_1 i_2 \ldots}$ represents the subset $\{\alpha_{i_1}, \alpha_{i_2}, \ldots\}$ of Δ fixed by ϱ which corresponds to the element in the Hasse diagram of $\Lambda_\sigma\backslash 0$, henceforth. So $\Lambda_\sigma\backslash 0$ is shown in Fig. 2, where $i_1 i_2 \ldots$, which equals $\pi(\overline{j_1 j_2 \ldots})$ for some $\overline{j_1 j_2 \ldots}$ in the Hasse diagram of $\Lambda\backslash 0$, represents the subset $\{\beta_{i_1}, \beta_{i_2}, \ldots\}$ of Δ_ϱ in the Hasse diagram of $\Lambda_\sigma\backslash 0$, henceforth.

(2) $\mu = k\mu_3$. In this case $J_0 = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6\}$ and $J_0' = \{\beta_1, \beta_2, \beta_4\}$. $\Lambda\backslash 0$ is shown in Fig. 3 and $\Lambda_\sigma\backslash 0$ is shown in Fig. 4.

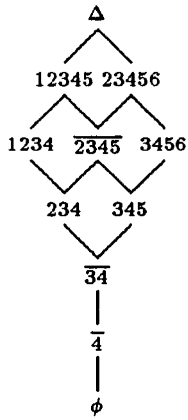


FIGURE 1

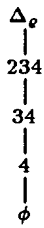


FIGURE 2

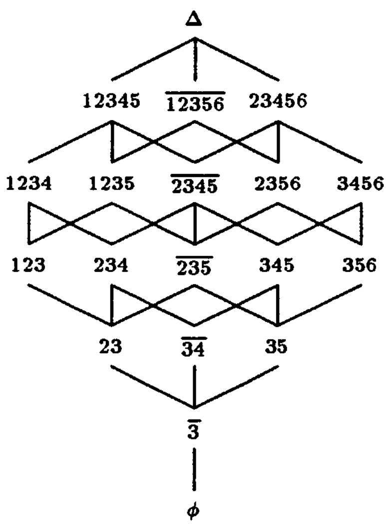


FIGURE 3

(3) $\mu = k_1 \mu_3 + k_2 \mu_4$. In this case $J_0 = \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\}$ and $J_0' = \{\beta_1, \beta_2\}$. $\Lambda \setminus 0$ is shown in Fig. 5 and $\Lambda_\sigma \setminus 0$ is shown in Fig. 6.

(4) $\mu = k\mu_5 + k\mu_2$. In this case $J_0 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$ and $J_0' = \{\beta_1, \beta_3, \beta_4\}$. $\Lambda \setminus 0$ is shown in Fig. 7 and $\Lambda_\sigma \setminus 0$ is shown in Fig. 8.

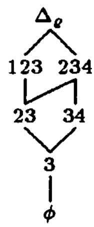


FIGURE 4

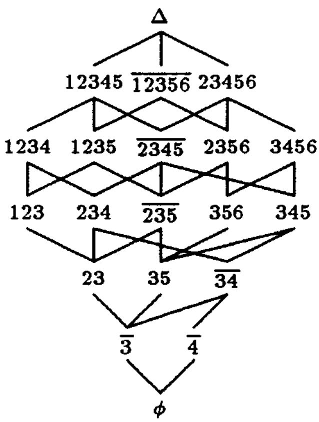


FIGURE 5

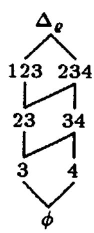


FIGURE 6

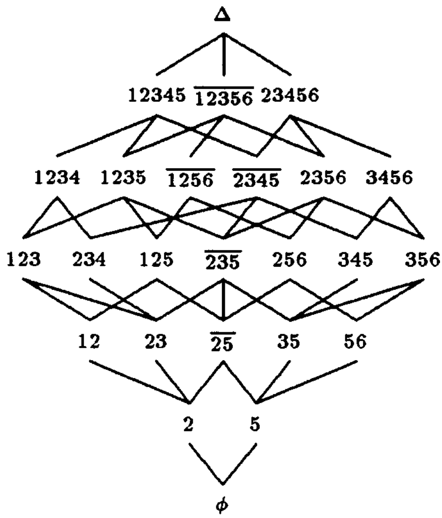


FIGURE 7

(5) $\mu = k\mu_1 + k\mu_6$. In this case $J_0 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and $J_0' = \{\beta_2, \beta_3, \beta_4\}$. $\Lambda\backslash 0$ is shown in Fig. 9 and $\Lambda_\sigma\backslash 0$ is shown in Fig. 10.

(6) $\mu = k_1\mu_3 + k\mu_2 + k\mu_5$. In this case $J_0 = \{\alpha_1, \alpha_4, \alpha_6\}$ and $J_0' = \{\beta_1, \beta_4\}$. $\Lambda\backslash 0$ is shown in Fig. 11 and $\Lambda_\sigma\backslash 0$ is shown in Fig. 12.

(7) $\mu = k_1\mu_4 + k\mu_2 + k\mu_5$. In this case $J_0 = \{\alpha_1, \alpha_3, \alpha_6\}$ and $J_0' = \{\beta_1, \beta_3\}$. $\Lambda\backslash 0$ is shown in Fig. 13 and $\Lambda_\sigma\backslash 0$ is shown in Fig. 14.

(8) $\mu = k_1\mu_3 + k\mu_1 + k\mu_6$. In this case $J_0 = \{\alpha_2, \alpha_4, \alpha_5\}$ and $J_0' = \{\beta_2, \beta_4\}$. $\Lambda\backslash 0$ is shown in Fig. 15 an $\Lambda_\sigma\backslash 0$ is shown in Fig. 16.

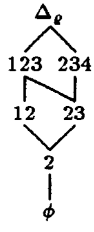


FIGURE 8

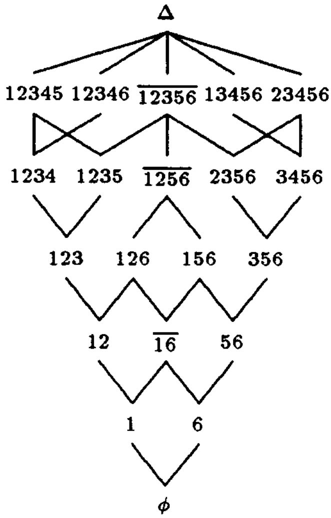


FIGURE 9

(9) $\mu = k_1 \mu_4 + k\mu_1 + k\mu_6$. In this case $J_0 = \{\alpha_2, \alpha_3, \alpha_5\}$ and $J_0' = \{\beta_2, \beta_3\}$. $\Lambda \backslash 0$ is shown in Fig. 17 and $\Lambda_\sigma \backslash 0$ is shown in Fig. 18.

(10) $\mu = k_1 \mu_3 + k_2 \mu_4 + k\mu_1 + k\mu_6$. In this case $J_0 = \{\alpha_2, \alpha_5\}$ and $J_0' = \{\beta_2\}$. $\Lambda \backslash 0$ is shown in Fig. 19 and $\Lambda_\sigma \backslash 0$ is shown in Fig. 20.

(11) $\mu = k_1 \mu_3 + k_2 \mu_4 + k\mu_2 + k\mu_5$. In this case $J_0 = \{\alpha_1, \alpha_6\}$ and $J_0' = \{\beta_1\}$. $\Lambda \backslash 0$ is shown in Fig. 21 and $\Lambda_\sigma \backslash 0$ is shown in Fig. 22.

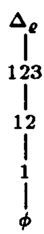


FIGURE 10

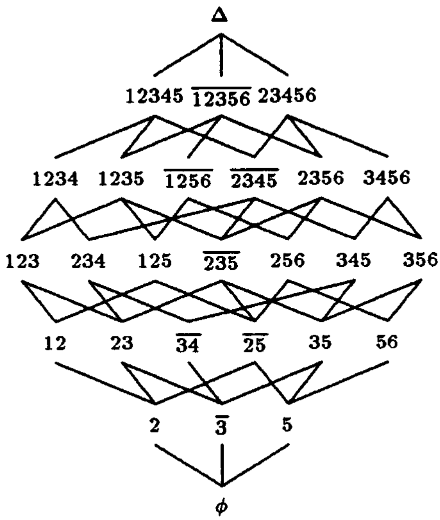


FIGURE 11

(12) $\mu = k_1\mu_1 + k_1\mu_6 + k\mu_2 + k\mu_5$. In this case $J_0 = \{\alpha_3, \alpha_4\}$ and $J_0' = \{\beta_3, \beta_4\}$. $\Lambda\backslash 0$ is shown in Fig. 23 and $\Lambda_\sigma\backslash 0$ is shown in Fig. 24.

(13) $\mu = k_1\mu_1 + k_1\mu_6 + k_2\mu_2 + k_2\mu_5 + k\mu_3$. In this case $J_0 = \{\alpha_4\}$ and $J_0' = \{\beta_4\}$. $\Lambda\backslash 0$ is shown in Fig. 25 and $\Lambda_\sigma\backslash 0$ is shown in Fig. 26.

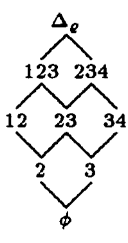


FIGURE 12

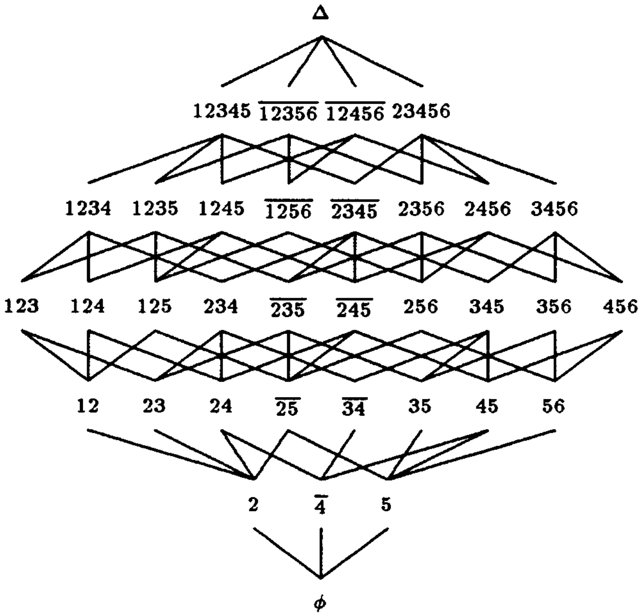


FIGURE 13

(14) $\mu = k_1\mu_1 + k_1\mu_6 + k_2\mu_2 + k_2\mu_5 + k\mu_4$. In this case $J_0\{\alpha_3\}$ and $J_0' = \{\beta_3\}$. $\Lambda \backslash 0$ is shown in Fig. 27 and $\Lambda_\sigma \backslash 0$ is shown in Fig. 28.

(15) $\mu = k_1\mu_1 + k_1\mu_6 + k_2\mu_2 + k_2\mu_5 + k_3\mu_3 + k_4\mu_4$. In this case $J_0 = \varnothing$ and $J_0' = \varnothing$. So $\Lambda \backslash 0 \cong \mathscr{P}(\Delta)$ and $\Delta_\sigma \backslash 0 \cong \mathscr{P}(\Delta_\sigma)$.

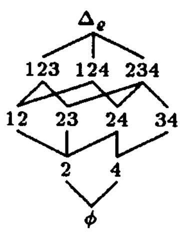


FIGURE 14

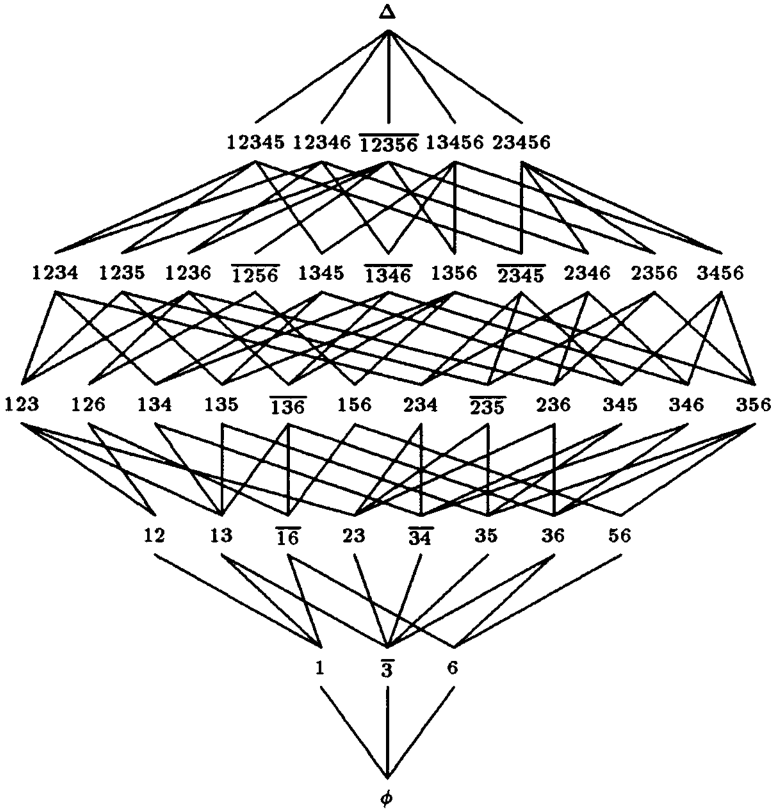


FIGURE 15

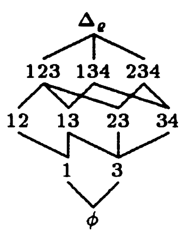


FIGURE 16

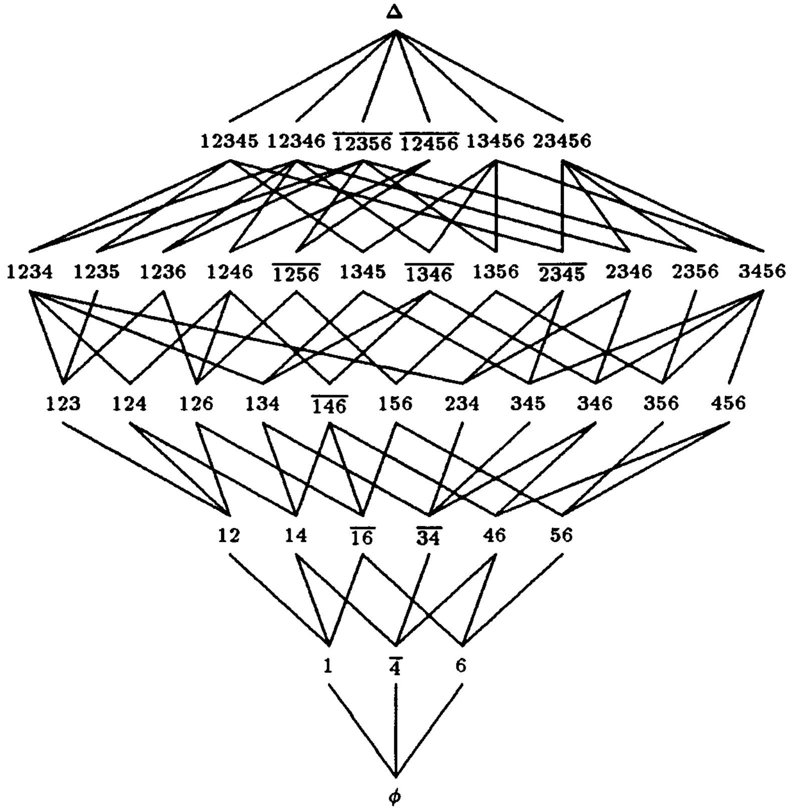


FIGURE 17

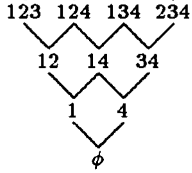


FIGURE 18

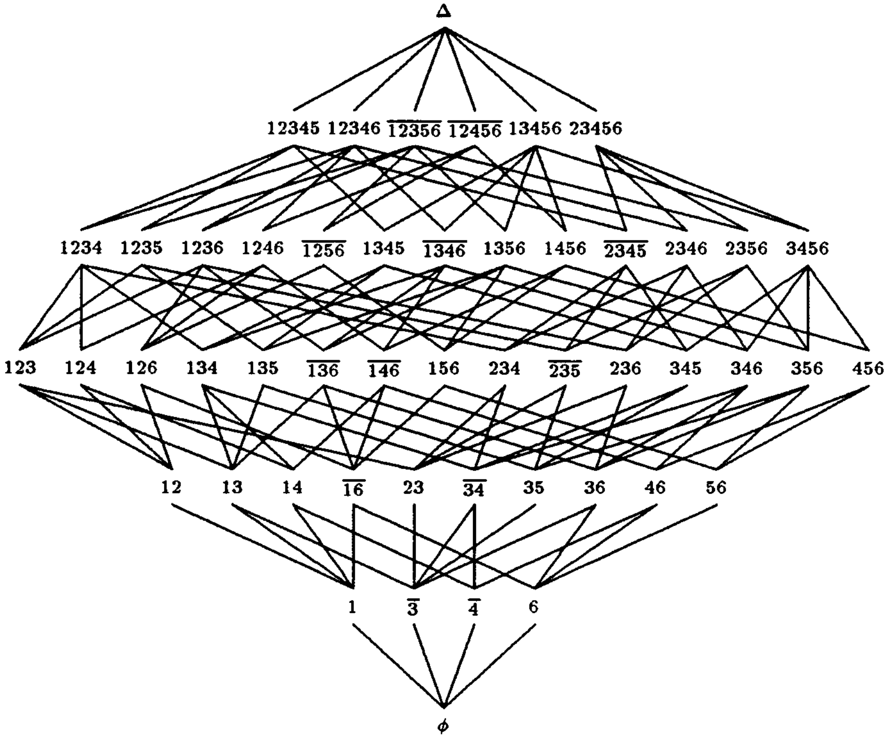


FIGURE 19

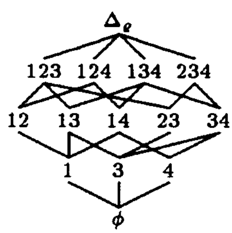


FIGURE 20

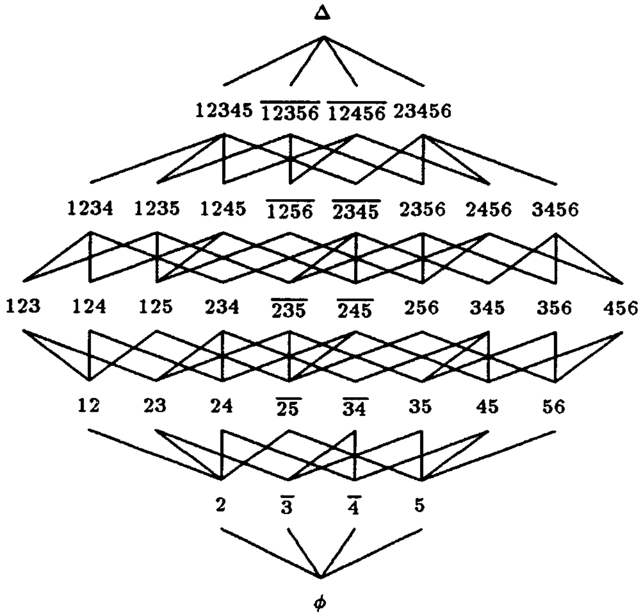


FIGURE 21

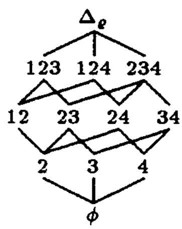


FIGURE 22

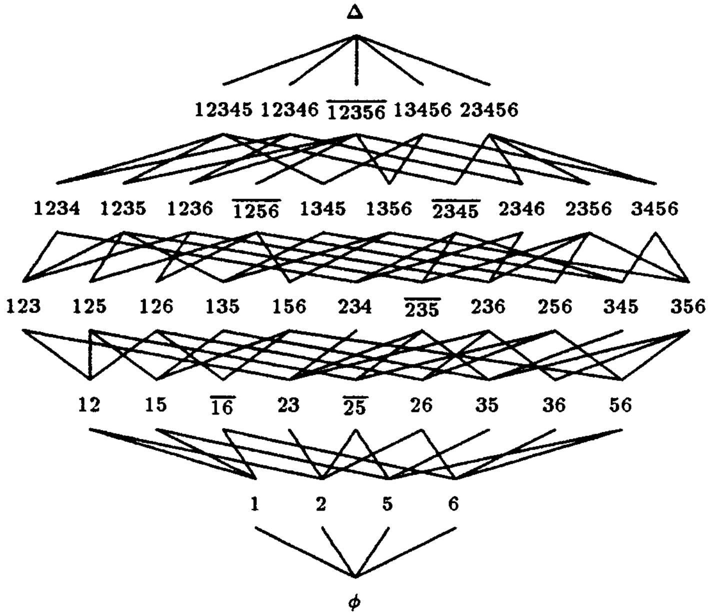


FIGURE 23

Remark. Theorem 2.2 contains a precise recipe to determine the structure of the cross section lattice for a $\mathscr{J}$-irreducible monoid. That recipe and Theorem 4.11 give a complete solution to (1) of 4.8. It is interesting but perhaps difficult to find a general answer for (2) of 4.8. Since M_σ is $\mathscr{J}$-irreducible we may expect a general recipe to determine the cross section lattice at least for Λ_σ.

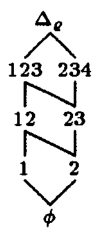


FIGURE 24

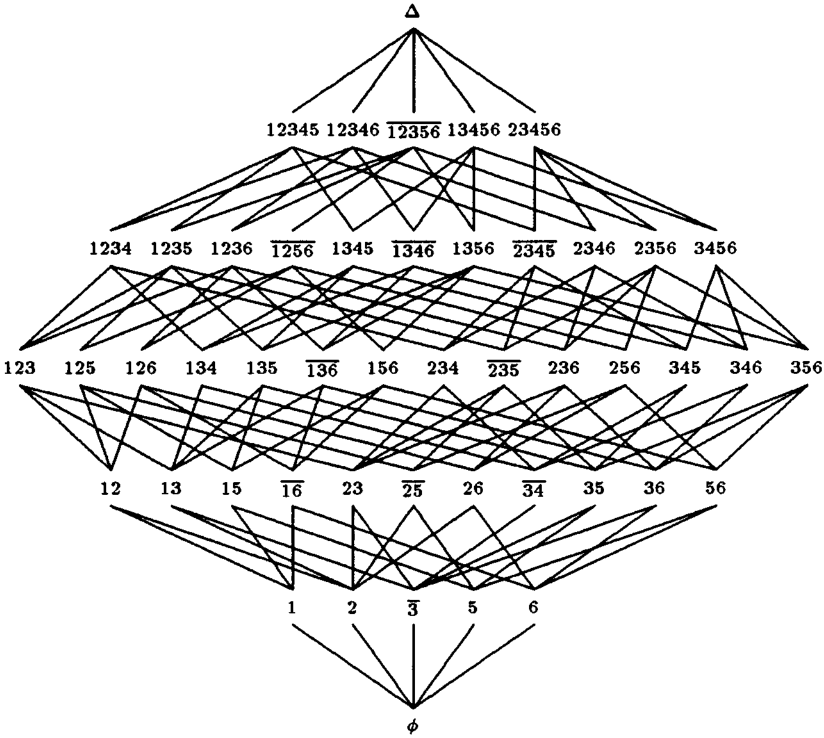


FIGURE 25

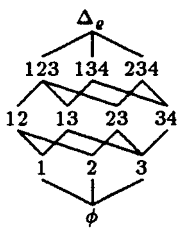


FIGURE 26

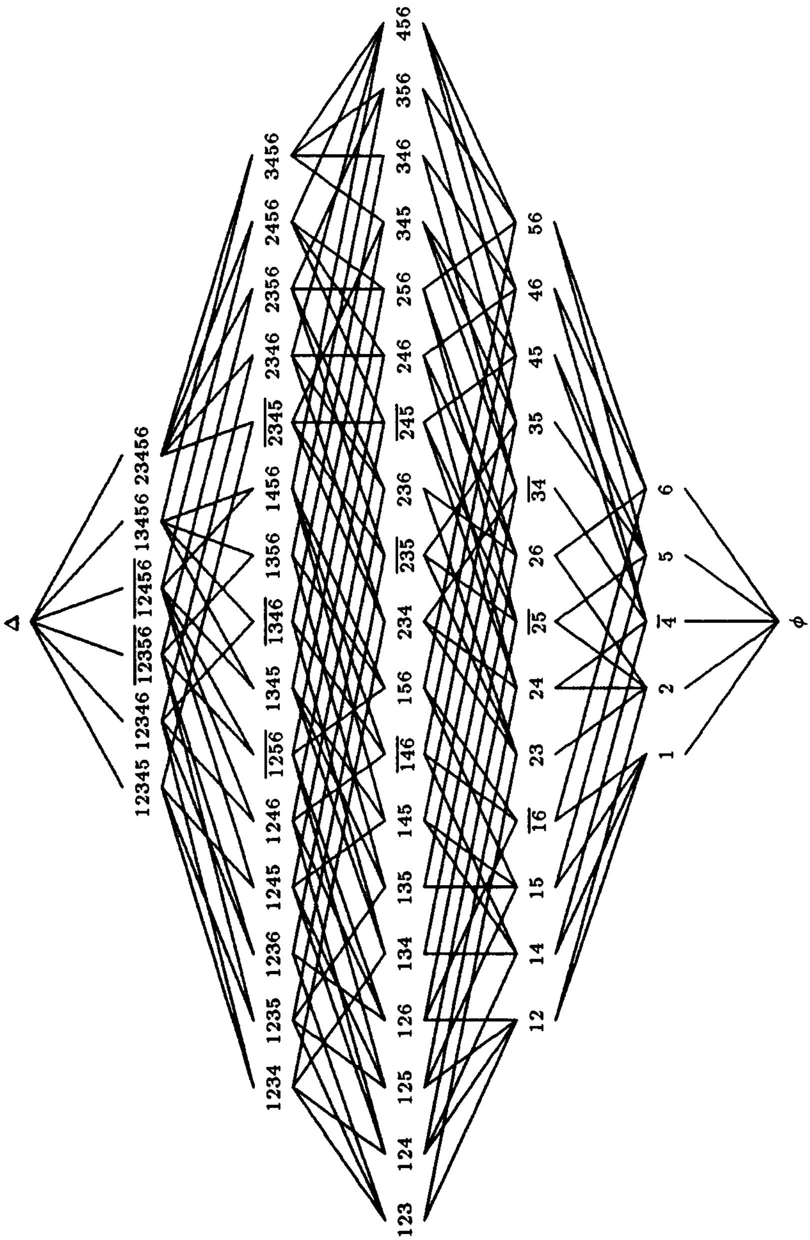

Δ
12345 12346 $\overline{12356}$ $\overline{12456}$ 13456 23456
1234 1235 1236 1245 1246 $\overline{1256}$ 1345 $\overline{1346}$ 1356 1456 $\overline{2345}$ 2346 2356 2456 3456
123 124 125 126 134 135 145 $\overline{146}$ 156 234 $\overline{235}$ 236 $\overline{245}$ 246 256 345 346 356 456
12 14 15 $\overline{16}$ 23 24 $\overline{25}$ 26 $\overline{34}$ 35 45 46 56
1 2 $\overline{4}$ 5 6
ϕ

FIGURE 27

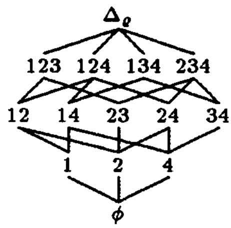


FIGURE 28

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